Kuwait University **Math 102** **Date: 03/05/2007**

Dept. of Math & Computer. Sci. (Second Midterm) Duration: 90 minutes

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(Answer all the following questions. Calculators and mobile phones are *not* allowed)

1. (3 points each) Evaluate the following integrals

a) $\int \frac{\sin^{-1}(e^x)}{e^{-x}} dx$

b) $\int \left(\frac{\sec x}{\tan x}\right)^4 dx$

c) $\int \frac{(\ln x)^3}{x\sqrt{4-(\ln x)^2}} dx$

d) $\int \frac{3x^2+1}{x^4-1} dx$

e) $\int \frac{\coth x}{\sqrt{\cosh^2 x-2}} dx$

2. (4 points) Determine whether the following integral is convergent or divergent, and find its value in the case of convergence.

$$\int_1^3 \frac{1}{\sqrt{3x-x^2}} dx.$$

3. (3 points each)

a) Find the following limit if it exists

$$\lim_{x\to\infty}\left(1-e^{-x}\right)^{e^x}$$

b) Find b such that

$$\lim_{x\to 0}\frac{\sin 3x-3x+bx^3}{x^3}=0$$

$$\text{(a)} \int \frac{\sin^{-1}(e^x)}{e^{-x}}dx$$

Take $t = e^x$, and use integration by parts to get $\int \frac{\sin^{-1}(e^x)}{e^{-x}}dx = e^x \sin(e^x) + \sqrt{1-e^{2x}} + C$.

$$\text{(b)} \int (\frac{\sec x}{\tan x})^4 dx$$

$\int (\frac{\sec x}{\tan x})^4 dx = \int \frac{\sec^2 x \sec^2 x}{\tan^4 x}dx = \int \frac{1+u^2}{u^4}$ (with $u = secx$ and using $1+\tan^2 x = \sec^2 x$.), So,

$$\int (\frac{\sec x}{\tan x})^4 dx = \frac{-1}{3}\tan^{-3} x - \tan^{-1} x + C.$$

$$\text{(c)} \int \frac{(\ln x)^3}{x\sqrt{4-(\ln x)^2}}dx$$

Take $t = \ln x$, and use trigonometric substitution $t = \sin(\alpha)$, to obtain

$$\int \frac{(\ln x)^3}{x\sqrt{4-(\ln x)^2}}dx = 8[-\cos(\alpha) + \frac{1}{3}\cos^3(\alpha)] + C$$

$$\text{(d)} \int \frac{3x^2+1}{x^4-1}dx$$

The decomposition in partial fractions gives

$$\frac{3x^2+1}{x^4-1} = \frac{1}{x-1} - \frac{1}{x+1} + \frac{1}{x^2+1}$$

Hence,

$$\int \frac{3x^2+1}{x^4-1}dx = \ln(x-1) - \ln(x+1) + \tan^{-1} x + C.$$

$$\text{(e)} \int \frac{\coth x}{\sqrt{\cosh^2 x - 2}}dx$$

$$\int \frac{\coth x}{\sqrt{\cosh^2 x - 2}} dx = \int \frac{\cosh x}{\sinh x \sqrt{\sinh^2 x - 1}} dx = \sec^{-1}(\sinh x) + C.$$

2. $3x - x^2 = \frac{9}{4} - (x - \frac{3}{2})^2$. Take $x - \frac{3}{2} = \frac{3}{2}\sin(\alpha)$ to obtain

$$\int_1^3 \frac{dx}{\sqrt{3x - x^2}} = \lim_{x \to 3^-} \sin^{-1}(\frac{2}{3}x - 1) - \sin^{-1}(\frac{2}{3} - 1) = \frac{\pi}{2} - \sin^{-1}(\frac{2}{3} - 1)$$

3. **(a)** $\lim_{x \to \infty} e^x \ln(1 - e^{-x}) = \lim_{x \to \infty} \frac{\ln(1-e^{-x})}{e^{-x}} = \lim_{x \to \infty} \frac{-1}{1-e^{-x}} = -1.$ $\lim_{x \to \infty}(1 - e^{-x})^{e^x} = \frac{1}{e}$.

(b) Applying l'Hospital's rule three times, we obtain

$$\lim_{x \to 0} \frac{\sin(3x) - 3x + bx^3}{x^3} = \frac{-27 + 6b}{6} = 0$$

So, $b = \frac{9}{2}$.